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The algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ and infinite Hopf family of algebras

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Abstract

New deformed affine algebras, $\mathcal{A}_{\hbar,\eta}(\hat{g})$, are defined for any simply laced classical Lie algebra g, which are generalizations of the algebra, $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$, recently proposed by Khoroshkin–Lebedev–Pakuliak (KLP). Unlike the work of KLP, we associate with the new algebras the structure of an infinite Hopf family of algebras in contrast to the one containing only finite number of algebras, introduced by KLP. Bosonic representation for $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at level 1 is obtained, and it is shown that, by repeated application of Drinfeld-like comultiplications, a realization of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at any positive integer level can be obtained. For the special case of $g = sl_{r+1}$, $(r+1)$-dimensional evaluation representation is given. The corresponding interwining operations are also discussed.

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1. Introduction

Since Drinfeld [9–11] proposed the quantum groups and Yangian algebras as deformations of the universal envoloping algebras of the classical Lie algebras, Hopf algebras with nontrivial coalgebra structure, especially q-affine algebras [32] and Yangian doubles [22,23], have become one of the major subjects of pure and applied algebra studies. Recently progress in the study of Hopf algebras and applications include the free boson representations of q-affine algebras and Yangian doubles at higher level [4,18,25], and the possibility of describing the dynamical symmetries and solving the correlation functions

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of certain solvable lattice statistic models and integrable quantum field theories within a pure algebraic framework [1,5,26–28,30,35]. The latter problem is, if not the sole force, among the driving forces which lead to the study of deformed algebras beyond Hopf algebras. Examples of such deformed algebras are q- [2,3,12,16,29–31,33,34] and $\hbar$- [7,17] deformed Virasoro and W algebras, the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ [14,15] and its scaling limit $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ [24], and the algebra of screening operators of the q-deformed W-algebras [13] and so on.

In this paper, we extend the recent work of Khoroshkin et al. [24] on the scaling algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ to the general case, $\mathcal{A}_{\hbar,\eta}(\hat{g})$, where g can be any classical simply laced Lie algebra of any admissible rank. The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ introduced in [24] is a formal algebra with generators carrying *continuous* indexes. One of the principal motivation of [24] was to establish a better understanding of the algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ from the representation theory point of view because the representation theory of $\mathcal{A}_{q,p}(\widehat{sl}_2)$ has been rather obscure since its birth [14,15]. For this, the authors of [24] considered the scaling limit $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$, instead of $\mathcal{A}_{q,p}(\widehat{sl}_2)$ itself, with generating functions being analytic along some strip – which plays the role of fundamental parallelogram for the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ – in the complex plane. The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ turns out to be *not* a Hopf algebra but belongs to a Hopf family of algebras in which the comultiplication can be made associative on changing the periods of structure functions for different iterations of the comultiplication. Moreover, the twisted intertwining operators appearing in the representation theory of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ satisfy a familar set of commutation relations which were used in the calculation of correlation functions for Sine–Gordon model.

We shall show that the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ actually belongs to (and constitutes the simplest example of) a new type of deformed affine algebras, $\mathcal{A}_{\hbar,\eta}(\hat{g})$. Just like their simplest representative, $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$, these new deformed affine algebras are not Hopf algebras, because the second deformation parameter η spoils the usual Hopf algebra structure. However, we regard them as *deformations* of the usual Hopf algebras for two reasons. First, as the second deformation parameter η approaches zero, the currents for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ obey the same commutation relations as that of the Yangian double with center $DY_\hbar(g)_c$. Second, if we consider the zero level representation of $\mathcal{A}_{\hbar,\eta}(\hat{g})$, the algebraic relations become Hopf algebra again.

Due to the complication for the case of general g, we restrict ourselves only to the current realization. In this form, it is not easy to write down the analog of comultiplication used by KLP [24]. We, therefore, use an analog of the well-known Drinfeld comultiplication to study some aspects in the structure of our algebra. It is, however, not known whether the finite Hopf family structure of KLP can be realized using this form of comultiplication. To circumvent this drawback, we introduce an alternative notion, which we call the infinite Hopf family of algebras, to write down the interactions of comultiplications in a convenient form. It turns out that this new notion leads to an astonishingly by simple realization of our algebra at any positive integer level.

Besides the pure algebraic elegance, our algebras are also expected to have relevant applications in such fields as an algebraic formulation of quantum symmetry and the calculation of correlation functions for affine Toda field theories.

2. The algebras $\mathcal{A}_{\hbar,\eta}(\hat{g})$ and infinite Hopf families

2.1. The algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$

We begin our study by introducing the formal current algebra (denoted also by the symbol $\mathcal{A}_{\hbar,\eta}(\hat{g})$) associated with $\mathcal{A}_{\hbar,\eta}(\hat{g})$. The special case of $g = sl_2$ can be inferred from [24]. For other simply laced classical algebras g, the following definition has, to our knowledge, not been introduced anywhere else.

Definition 1. The current algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ associated with the classical simply laced Lie algebra g of rank r (as an associative algebra with unit over the field $\mathbb{C}$) is generated by the $3r$ currents $\{H_i^{\pm}(u), E_i(u), F_i(u) | i = 1, \ldots, r\}$, the central element c and 1 with the following generating relations: [1]

$$\begin{aligned} &H_i^{\pm}(u)H_j^{-}(v) \\ &\quad = \frac{\sinh \pi\eta(u - v - i\hbar(B_{ij} - c/2))}{\sinh \pi\eta(u - v + i\hbar(B_{ij} + c/2))} \frac{\sinh \pi\eta'(u - v + i\hbar(B_{ij} - c/2))}{\sinh \pi\eta'(u - v - i\hbar(B_{ij} + c/2))} \\ &\quad \times H_j^{-}(v)H_i^{+}(u), \end{aligned} \tag{1}$$

$$\begin{aligned} H_i^{\pm}(u)H_j^{\pm}(v) = &\frac{\sinh \pi\eta(u - v - i\hbar B_{ij})}{\sinh \pi\eta(u - v + i\hbar B_{ij})} \frac{\sinh \pi\eta'(u - v + i\hbar B_{ij})}{\sinh \pi\eta'(u - v - i\hbar B_{ij})} \\ &\times H_j^{\pm}(v)H_i^{\pm}(u), \end{aligned} \tag{2}$$

$$H_i^{\pm}(u)E_j(v) = \frac{\sinh \pi\eta(u - v - i\hbar(B_{ij} \mp c/4))}{\sinh \pi\eta(u - v + i\hbar(B_{ij} \pm c/4))} E_j(v)H_i^{\pm}(u), \tag{3}$$

$$H_i^{\pm}(u)F_j(v) = \frac{\sinh \pi\eta'(u - v + i\hbar(B_{ij} \mp c/4))}{\sinh \pi\eta'(u - v - i\hbar(B_{ij} \pm c/4))} F_j(v)H_i^{\pm}(u), \tag{4}$$

$$E_i(u)E_j(v) = \frac{\sinh \pi\eta(u - v - i\hbar B_{ij})}{\sinh \pi\eta(u - v + i\hbar B_{ij})} E_j(v)E_i(u), \tag{5}$$

$$F_i(u)F_j(v) = \frac{\sinh \pi\eta'(u - v + i\hbar B_{ij})}{\sinh \pi\eta'(u - v - i\hbar B_{ij})} F_j(v)F_i(u), \tag{6}$$

$$\begin{aligned} [E_i(u), F_j(v) = &\frac{2\pi}{\hbar}\delta_{ij} \left[\delta\left(u - v - \frac{i\hbar c}{2}\right) H_i^{+}\left(u - \frac{i\hbar c}{4}\right) \right. \\ &\left. - \delta\left(u - v + \frac{i\hbar c}{2}\right) H_i^{-}\left(v - \frac{i\hbar c}{4}\right) \right], \end{aligned} \tag{7}$$

$$\begin{aligned} &E_i(u_1)E_i(u_2)E_j(v) - 2\cos(\pi\eta\hbar)E_i(u_1)E_j(v)E_i(u_2) + E_j(v)E_i(u_1)E_i(u_2) \\ &\quad + (u_1 \leftrightarrow u_2) = 0, \quad \text{for } A_{ij} = -1, \end{aligned} \tag{8}$$

[1] Throughout this paper, the subscript i of the current operators take integral values, which indicates different root directions of the underlying Lie algebra g, whilst the symbol i preceding the $\hbar$'s in the structure functions represents $\sqrt{-1}$.

$$F_i(u_1)F_i(u_2)F_j(v) - 2\cos(\pi\eta'\hbar)F_i(u_1)F_j(v)F_i(u_2) + F_j(v)F_i(u_1)F_i(u_2)$$
$$+(u_1 \leftrightarrow u_2) = 0, \quad \text{for } A_{ij} = -1, \tag{9}$$
$$[c, \text{everything}] = 0, \tag{10}$$

where u, v etc. are spectral parameters, real $\hbar$, η are two deformation parameters, $B_{ij} = A_{ij}/2$, A_{ij} are matrix elements of the Cartan matrix for the Lie algebra g, and [2]

$$\frac{1}{\eta'} - \frac{1}{\eta} = \hbar c.$$

Remark 1. For $g = sl_2$, the above current algebra reduces to the current realization of $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$, where the Serre-like relations (8) and (9) are not present.

Remark 2. In the limit $\eta \to 0$, the current algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ would have the same form as that of the Yangian double $DY_\hbar(g)$. But the limiting algebra $\mathcal{A}_{\hbar,0}(\hat{g})$ should not be considered to be isomorphic with the Yangian double $DY_\hbar(g)$ because the element of the algebra $\mathcal{A}_{\hbar,0}(\hat{g})$ carries a continuous index whilst that of the Yangian double $DY_\hbar(g)$ carries a discrete one. For more information on $g = sl_2$, see [24].

To have a precise definition for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ (and *not* its current realization form), we have to consider two different cases as in [24] for sl_2: (1) the case $c \neq 0$; and (2) the case $c = 0$. In the first case, one should consider the current $\{H_i^\pm(u), E_i(u), F_i(u)\}$ as the following Fourier transforms of the actual elements $t_i(\lambda)$, $e_i(\lambda)$ and $f_i(\lambda)(\lambda \in \mathbb{R})$ of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$,

$$H_i^\pm(u) = -\frac{\hbar}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} t_i(\lambda)\mathrm{e}^{\mp\lambda/2\eta''}, \quad \eta'' = \frac{2\eta\eta'}{\eta+\eta'},$$

$$E_i(u) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u}(\lambda),$$

$$F_i(u) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} f_i(\lambda),$$

whereas in the second case, the currents $H_i^\pm(u)$ should be given another expression in terms of the actual elements $h_i(\lambda)$ of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at $c = 0$,

$$H_i^\pm(u) = \mp\hbar \int_{-\infty}^{\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} \frac{h_i(\lambda)}{1 - \mathrm{e}^{\mp\lambda/\eta}}.$$

[2] We assume throughout this paper that η and $\hbar$ are generic, i.e. $\hbar$ is not a rational multiple of η.

The difference between $c \neq 0$ and $c = 0$ can be summarized in a more compact relationship between the algebra generators $t_i(\lambda)$ and $h_i(\lambda)$. In fact, from the two expressions of $H_i^{\pm}(u)$, we can write down the following relation at $c = 0$,

$$h_i(\lambda) = t_i(\lambda)\sinh\left(\frac{\lambda}{2\eta}\right).$$

Therefore, in the limit $c \to 0$, $h_i(0)$ is well-defined, but $t_i(0)$ tends to infinity. On the contrary, when $c \neq 0$, $t_i(0)$ is well-defined and $h_i(0)$ tends to zero.

Given the above Fourier transformations, one can, in principle, write down the generating relations for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ in terms of the continuous generators $t_i(\lambda)(h_i(\lambda))$, $e_i(\lambda)$ and $f_i(\lambda)$. However, such relations are rather complicated and they are of no use in the rest of this paper. Therefore, we shall omit such relations and consider only the *current realization* (1)–(10) of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$.

Unlike the usual q-affine algebras and the Yangian doubles which are by definition non-trivial Hopf algebras, whether the algebra under consideration has a Hopf algebra structure is not known. Recall that a Hopf algebra $\mathcal{A}$ is an algebra endowed with five operations (linear maps):

- the algebra multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $m(a \otimes b) = ab$ for $\forall a, b \in \mathcal{A}$;
- the unit embedding $\iota : \mathbb{C} \to \mathcal{A}$, $\iota(c) = c1$, $c \in \mathbb{C}$, $1 \in \mathcal{A}$ is the unit element;
- comultiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\Delta(ab) = \Delta(a)\Delta(b)$ for $\forall a, b \in \mathcal{A}$;
- the antipode $S : \mathcal{A} \to \mathcal{A}$, $S(ab) = S(b)S(a)$ for $\forall a, b \in \mathcal{A}$; and
- the counit $\epsilon : \mathcal{A} \to \mathbb{C}$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$, for $\forall a \in \mathcal{A}$, where $\epsilon(a) \in \mathbb{C}$.

To make the algebra $\mathcal{A}$ into a Hopf algebra, these structures have to obey the following axioms,

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m), \tag{11}$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{12}$$

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \tag{13}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta. \tag{14}$$

For our algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$, only the first of these axioms holds explicitly, which ensures the associativity of the algebra multiplication. The operations Δ, ϵ, S cannot be defined on the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ alone. However, as first discovered in [24], a well-defined coproduct can be defined over the so-called "Hopf family of algebra" containing a finite number of algebras of the kind $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$ but with different parameters η. However, as stated in the introduction, the case for arbitrary g is much more complicated and we can only make our analysis in the current realization. This difficulty prevented us from obtaining an analogous structure of KLP's Hopf family of algebras because the analogous comultiplication is not known. Therefore, we proceed to introduce an alternative notion – the infinite Hopf family of algebras. it should be remarked that no relationship is implied here between our infinite Hopf family of algebras and the (finite) Hopf family of algebras introduced by KLP.

Definition 2. Let $\{A_n, n \in \mathbb{Z}\}$ be a family of associative algebras with unit defined over $\mathbb{C}$. If on each $\mathcal{A}_n$ one can define the following operations

- the comultiplications $\Delta_n^+ : \mathcal{A}_n \to \mathcal{A}_n \otimes \mathcal{A}_{n+1}$, $\Delta_n^+(a_{(n)}) = \sum_i a'_{(n)i} \otimes a'_{(n+1)i}$ and $\Delta_n^- : \mathcal{A}_n \to \mathcal{A}_{n-1} \otimes \mathcal{A}_n$, $\Delta_n^-(a_{(n)}) = \sum_i a'_{(n-1)i} \otimes a'_{(n)i}$, where $a_{(n)i}, a'_{(n)i}, etc. \in \mathcal{A}_n$ and $\Delta_n^\pm$ are algebra morphisms;
- the counits $\epsilon_n : \mathcal{A}_n \to \mathbb{C}$; and
- the antipodes $S_n^\pm : \mathcal{A}_n \to \mathcal{A}_{n\pm 1}$, $S_n^\pm(a_{(n)}b_{(n)}) = S_n^\pm(b_{(n\pm 1)})S_n^\pm(a_{(n\pm 1)})$, which are anti-morphisms,

and if they satisfy the following constraints,

$$(\epsilon_n \otimes \mathrm{id}_{n+1})\Delta_n^+ = \mathrm{id}_{n+1}, \tag{15}$$

$$(\mathrm{id}_{n-1} \otimes \epsilon_n)\Delta_n^- = \mathrm{id}_{n-1}, \tag{16}$$

$$m_{n+1} \circ (S_n^+ \otimes \mathrm{id}_{n+1}) \circ \Delta_n^+ = \epsilon_{n+1}, \tag{17}$$

$$m_{n-1} \circ (\mathrm{id}_{n-1} \otimes S_n^-) \circ \Delta_n^- = \epsilon_{n-1}, \tag{18}$$

where m_n is the algebra multiplication for the nth component algebra $\mathcal{A}_n$, then we call the family of algebras $\{\mathcal{A}_n, n \in \mathbb{Z}\}$ an infinite Hopf family of algebras.

A trivial example for the infinite Hopf family of algebras is the family $\{\mathcal{A}_n \equiv \mathcal{A}, n \in \mathbb{Z}\}$ in which $\mathcal{A}$ is a usual Hopf algebra. In this case, all our axioms (15)–(18) hold with the comultiplications $\Delta_n^\pm$, counits ϵ_n and the antipodes $S_n^\pm$ being identified with those corresponding structures for the usual Hopf algebra. Note that, in this trivial case, we have one more axiom, Eq. (12), which represents the coassociativity of the comultiplication. For the general case, no coassociativity is required. One may consider the lack of coassociativity in our infinite Hopf family of algebras a serious drawback compared to the (finite) Hopf family structure of [24]. However, it will soon be clear in Proposition 2 that this structure would bring about a great advantage in obtaining realizations of our algebra at integer levels $k > 1$.

We now proceed to construct a nontrivial example for the finite Hopf family of algebras containing our algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ as a member.

Let $\eta^{(0)} = \eta$. For all $n \in \mathbb{Z}$, let us define $\eta^{(n)}$ recursively, such that

$$\frac{1}{\eta^{(n+1)}} - \frac{1}{\eta^{(n)}} = \hbar c_n,$$

where c_n are a set of parameters and $c_0 \equiv c$, the center of our algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$. Clearly, for $n = 0$, we have $\eta^{(1)} = \eta'$. The notations $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n}$ have obvious meaning with the specification $\mathcal{A}_{\hbar,\eta^{(0)}}(\hat{g})_{c_0} = \mathcal{A}_{\hbar,\eta}(\hat{g})$.

Proposition 1. *The family of algebras $\{\mathcal{A}_n \equiv \mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n}, n \in \mathbb{Z}\}$ form an infinite Hopf family of algebras with the comultiplications $\Delta_n^\pm$, counits ϵ_n and antipodes $S_n^\pm$ defined as follows:*

- *the comultiplications $\Delta_n^\pm$:*

$$\Delta_n^+ c_n = c_n + c_{n+1}, \tag{19}$$

$$\Delta_n^- c_n = c_{n-1} + c_n, \tag{20}$$

$$\Delta_n^+ H_i^+(u;\eta^{(n)}) = H_i^+\left(u+\frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n)}\right)\otimes H_i^+\left(u-\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n+1)}\right), \tag{21}$$

$$\Delta_n^- H_i^+(u;\eta^{(n)}) = H_i^+\left(u+\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n-1)}\right)\otimes H_i^+\left(u-\frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n)}\right), \tag{22}$$

$$\Delta_n^+ H_i^-(u;\eta^{(n)}) = H_i^-\left(u-\frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n)}\right)\otimes H_i^-\left(u+\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n+1)}\right), \tag{23}$$

$$\Delta_n^- H_i^-(u;\eta^{(n)}) = H_i^-\left(u-\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n-1)}\right)\otimes H_i^-\left(u+\frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n)}\right), \tag{24}$$

$$\Delta_n^+ E_i(u;\eta^{(n)}) = E_i(u;\eta^{(n-1)})\otimes 1 + H_i^-\left(u+\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n)}\right)\otimes E_i\left(u+\frac{\mathrm{i}\hbar c_n}{2};\eta^{(n+1)}\right), \tag{25}$$

$$\Delta_n^- E_i(u;\eta^{(n)}) = E_i(u;\eta^{(n-1)})\otimes 1 + H_i^-\left((u+\frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n-1)}\right)\otimes E_i\left(u+\frac{\mathrm{i}\hbar c_{n-1}}{2};\eta^{(n)}\right), \tag{26}$$

$$\Delta_n^+ F_i(u;\eta^{(n)}) = 1\otimes F_i(u;\eta^{(n+1)}) + F_i\left(u+\frac{\mathrm{i}\hbar c_{n+1}}{2};\eta^{(n)}\right)\otimes H_i^+\left(u+\frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n+1)}\right), \tag{27}$$

$$\Delta_n^- F_i(u;\eta^{(n)}) = 1\otimes F_i(u;\eta^{(n)}) + F_i\left(u+\frac{\mathrm{i}\hbar c_n}{2};\eta^{(n-1)}\right)\otimes H_i^+\left(u+\frac{\mathrm{i}\hbar c_n}{4};\eta^{(n)}\right); \tag{28}$$

– *the counits ϵ_n:*

$$\epsilon_n(c_n)=0,\qquad \epsilon_n(1_n)=1,\qquad \epsilon_n(H_i^\pm(u;\eta^{(n)}))=1,$$
$$\epsilon_n(E_i(u;\eta^{(n)}))=0,\qquad \epsilon_n(F_i(u;\eta^{(n)}))=0;$$

– *the antipodes $S_n^\pm$:*

$$S_n^\pm(c_n) = -c_{n\pm1},$$
$$S_n^\pm(H_i^\pm(u;\eta^{(n)})) = [H_i^\pm(u;\eta^{(n\pm1)})]^{-1},$$
$$S_n^\pm(E_i(u;\eta^{(n)})) = -H_i^-\left(u-\frac{\mathrm{i}\hbar c_{n\pm1}}{4};\eta^{(n\pm1)}\right)^{-1} E_i\left(u-\frac{-\mathrm{i}\hbar c_{n\pm1}}{2};\eta^{(n\pm1)}\right),$$
$$S_n^\pm(F_i(u;\eta^{(n)})) = -F_i\left(u-\frac{\mathrm{i}\hbar c_{n\pm1}}{2};\eta^{(n\pm1)}\right) H_i^+\left(u-\frac{\mathrm{i}\hbar c_{n\pm1}}{4};\eta^{(n\pm1)}\right)^{-1}.$$

where the second arguments in the current operators (the η's) indicate to which algebra the currents belong.

The proof for this proposition is straightforward. Note that, in this example, the result of the action of the comultiplications Δ_{n-1}^{+} on $\mathcal{A}_{n-1}$ coincides with the result of the action of Δ_n^{-} on $\mathcal{A}_n$,

$$\Delta_n^{-}\mathcal{A}_n = \Delta_{n-1}^{+}\mathcal{A}_{n-1}.$$

Consequently, both comultiplications "co-commute",

$$(\mathrm{id} \otimes \Delta_n^{+})\Delta_n^{-} = (\Delta_n^{-} \otimes \mathrm{id})\Delta_n^{+}.$$

Two more remarks are in order.

Remark 3. In the case of $c_n = 0$, for all $n \in \mathbb{Z}$, the infinite Hopf family of algebras become trivial again because there are no differences between the algebras $\mathcal{A}_{\hbar\eta^{(n)}}(\hat{g})_0$ and $\mathcal{A}_{\hbar,\eta^{(m)}}(\hat{g})_0$ for any pair of $n, m \in \mathbb{Z}$.

Remark 4. Under the cases of Remarks 2 and 3, the above structures for the infinite Hopf family of algebras reduce to the original Hopf algebra structure. In particular, under the case of Remark 2, the comultiplications would have the same form with the so-called Drinfeld comultiplication for the Yangian double.

The comultiplications introduced above are useful not only in clarifying the structure of the infinite Hopf family of algebras but also in the representation theory of the representative algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$. Before going into detailed structure of representations, we state the following proposition, which can be directly verified.

Proposition 2. *The comultiplication Δ_n^{+} defined in Eqs. (19)–(28) induces algebra homomorphism from $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n+c_{n+1}}$ to $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n} \otimes \mathcal{A}_{\hbar,\eta^{(n+1)}}(\hat{g})_{c_{n+1}}$, Δ_n^{-} induces homomorphism from $\mathcal{A}_{\hbar,\eta^{(n-1)}}(\hat{g})_{c_{n-1}+c_n}$ to $\mathcal{A}_{\hbar,\eta^{(n-1)}}(\hat{g})_{c_{n-1}} \otimes \mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n}$.*

Actually, the above proposition states that the images of the generating currents $E_i(u; \eta^{(n)})$, $F_i(u; \eta^{(n)})$ and $H_i^{\pm}(u; \eta^{(n)})$ of $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n}$ under $\Delta_n^{\pm}$ satisfy the defining relations for $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_{c_n+c_{n+1}}$ and $\mathcal{A}_{\hbar,\eta^{(n-1)}}(\hat{g})_{c_{n-1}+c_n}$, respectively. This result is quite astonishing an one hand, and will be quite useful for constructing a higher level realization out of level 1 representations, on the other. Therefore, we proceed to consider the level 1 representation of our algebra.

3. Representation theory

3.1. *Free boson realization of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at level $c = 1$*

First, we would like to consider the free-boson realization of the generating relations (1)–(10) for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$. For this we introduce the set of deformed free bosons

Fig. 1. The integration contour C.

$a_i(\lambda)$ with continuous parameter $\lambda \neq 0$ and discrete $i = 1, \ldots, r$, which constitute the following deformed Heisenberg algebra $\mathcal{H}(\eta)$:

$$[a_i(\lambda), a_j(\mu)] = \frac{4}{\lambda} \sinh \frac{\hbar\lambda}{2} \sinh(\hbar B_{ij}\lambda) \frac{\sinh(\lambda/2\eta)}{\sinh(\lambda/2\eta')} \delta(\lambda + \mu). \tag{29}$$

We also use the notations $a_i'(\lambda) = a_i(\lambda)\sinh(\lambda/2\eta)/\sinh(\lambda/2\eta')$, which satisfy the relations

$$[a_i'(\lambda), a_j'(\mu)] = \frac{4}{\lambda} \sinh \frac{\hbar\lambda}{2} \sinh(\hbar B_{ij}\lambda) \frac{\sinh(\lambda/2\eta')}{\sinh(\lambda/2\eta)} \delta(\lambda + \mu).$$

The normal ordering for the exponential expressions of the above free bosons are defined in the following way [24],

$$\begin{aligned} &\exp \int_{-\infty}^{\infty} d\lambda g_1(\lambda) a_i(\lambda) \exp \int_{-\infty}^{\infty} d\mu g_2(\mu) a_j(\mu) \\ &= \exp \left(\int_C \frac{d\lambda \ln(-\lambda)}{2\pi i} \alpha_{ij}(\lambda) g_1(\lambda) g_2(-l) \right) \\ &\quad \exp \left(\int_{-\infty}^{\infty} d\lambda g_1(\lambda_1) a_i(\lambda) + \int_{-\infty}^{\infty} d\mu g_2(\mu) a_j(\mu) \right), \end{aligned} \tag{30}$$

where $\alpha_{ij}(\lambda)$ is a function given by

$$[a_i(\lambda), a_j(\mu)] = \alpha_{ij}(\lambda)\delta(\lambda + \mu), \tag{31}$$

C is a contour on the complex λ-plane depicted in Fig. 1. Moreover, we introduce the following zero mode operators,

$$[P_i, Q_j] = B_{ij}.$$

Proposition 3. *The following bosonic expressions realize the generating relations (1)–(10) of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ with $c = 1$,*

$$E_j(u) = e^{\gamma} \exp(2\pi i Q_j) \exp(P_j) \exp\left(\tfrac{1}{2}\phi_j'(u)\right), \tag{32}$$

$$F_j(u) = e^{\gamma} \exp(-2\pi i Q_j) \exp(-P_j) \exp(-\tfrac{1}{2}\phi_j(u)), \tag{33}$$

$$H_j^{\pm}(u) = \mathrm{e}^{-2\gamma} E_i \left(u \pm \frac{\mathrm{i}\hbar}{4}\right) F_j \left(u \mp \frac{\mathrm{i}\hbar}{4}\right) \tag{34}$$

$$= \exp\left(\mp \int_{-\infty}^{\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} \frac{\mathrm{e}^{\mp\hbar\lambda/4}}{1-\mathrm{e}^{\pm\lambda/\eta}} a_j(\lambda)\right), \tag{35}$$

where

$$\phi_j(u) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} \frac{a_j(\lambda)}{\sinh(\hbar\lambda/2)}, \tag{36}$$

$$\phi'_j(u) = \int_{-\infty}^{+\infty} \mathrm{d}\lambda \mathrm{e}^{\mathrm{i}\lambda u} \frac{a'_j(\lambda)}{\sinh(\hbar\lambda/2)}. \tag{37}$$

The proof of this proposition is also straightforward but requires tedious calculations. The normal ordering rule (30) and the following formula which can be found in [24] are very useful for the calculations,

$$\int_C \frac{\mathrm{d}\lambda \ln(-\lambda)}{2\pi \mathrm{i}\lambda} \frac{\mathrm{e}^{-x\lambda}}{1-\mathrm{e}^{-\lambda/\eta}} = \ln \Gamma(\eta x) + \left(\eta x - \frac{1}{2}\right)(\gamma - \ln \eta) - \tfrac{1}{2}\ln(2\pi),$$

where $\Gamma(x)$ is the usual Gamma function which satisfy the following,

$$\Gamma(x)\Gamma(1-x) = \frac{x}{\sin \pi x}.$$

It is interesting to mention that the bosonization formulas for the currents $E_i(u)$ and $F_i(u)$ are quite similar to those of the screening currents of the quantum $(\hbar, \xi)$-deformed W-algebras [17].

3.2. Representations at other integer levels

The bosonic expressions (32)–(37) only give a bosonic realization of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at level $c = 1$. However, as mentioned in Section 3.1, it is possible to obtain realizations of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ at other integer levels using the knowledge gathered so far. The key point is to make use of Proposition 2 repeatedly; first in the case of $c = c_0 = c_1 = 1$ (which leads to a realization at level $c = 2$), then in the case of $c = c_0 = 2, c_1 = 1$, and so on.

We give the following proposition

Proposition 4. *The level $c = k (k \in \mathbb{Z}_+)$ bosonic realization for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ can be obtained using k copies of the Heisenberg algebra (29) $\{\mathcal{H}(\eta^{(l)}), l = 0, 1, \ldots, k-1\}$ (each of which realizes the level 1 representation for the algebras $\mathcal{A}_{\hbar,\eta^{(l)}}(\hat{g})$ with $l = 0, 1, \ldots, k-1$) and the repeated use of Proposition 2 (the comultiplication Δ_0^+).*

Actually, the above proposition provides a way to understand the meaning of the infinite Hopf family of algebras – instead of getting higher level representations of any distinguished member of this family, one can study the level 1 representations for several members simultaneously. Though the resulting higher level representations obtained from the above iteration of comultiplications may be highly reducible, it is rather instructive to show the unusual way in which the tensor product representation is defined in our infinite Hopf family of algebras, i.e. one must obtain the tensor product of the representations of *different* members of the family.

To obtain bosonic realizations of negative integer level, one may use the antipodes $S_n^{\mp}$. However, such realizations are not of much interest to us.

3.3. The structure of Fock spaces

The bosonic realizations of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ can be viewed as representations on the Fock space of the bosonic Heisenberg algebras. Therefore, for completeness, we have to say some words on the structure of Fock spaces.

First, we specify the Fock space for the level 1 representation. Consider the abbreviated form (31) of the bosonic Heisenberg algebra $\mathcal{H}(\eta)$. The structure functions $\alpha_{ij}(\lambda)$ have the following properties:

$$\alpha_{ij}(\lambda) = -\alpha_{ij}(-\lambda), \qquad \alpha_{ij}(\lambda) = \alpha_{ji}(\lambda), \tag{38}$$

Let $|\mathrm{vac}\rangle$ be a right "vacuum state". The right Fock space $\mathcal{F}(\eta)$ is generated from $|\mathrm{vac}\rangle$ as follows:

$$\int_{-\infty}^{0} \mathrm{d}\lambda_n f_n(\lambda_n) a_{i_n}(\lambda_n) \cdots \int_{-\infty}^{0} \mathrm{d}\lambda_1 f_1(\lambda_1) a_{i_1}(\lambda_1) |\mathrm{vac}\rangle, \quad i_l = 1, 2, \ldots, \quad r, \quad l = 1, 2, \ldots, n,$$

where $f_l(\lambda)$ are functions which are analytic in a neighborhood of $\mathbb{R}_+$ except $\lambda = 0$, where a simple pole may appear. For each concrete $\alpha_{ij}(\lambda)$, proper asymptotic behaviors for $f_l(\lambda)$ as $\lambda \to +\infty$ are required. However, we do not specify them in detail (for the special case of $g = sl_2$, such asymptotics were given explicitly in [24]).

Similarly, let $\langle\mathrm{vac}|$ be a left "vacuum state". The left Fock space $\mathcal{F}^*(\eta)$ is generated as follows,

$$\langle\, \mathrm{vac}\,| \int_{0}^{+\infty} \mathrm{d}\lambda_1 g_1(\lambda_1) a_{i_1}(\lambda_1) \cdots \int_{0}^{+\infty} \mathrm{d}\lambda_n g_n(\lambda_n) a_{i_n}(\lambda_n), \quad i = 1, 2, \ldots, r, \quad l = 1, 2, \ldots n,$$

where $g_l(\lambda)$ are functions which are analytic in a neighborhood of $\mathbb{R}_-$ except $\lambda = 0$, where a simple pole may appear. As in the case of $f_l(\lambda)$, proper asymptotic behaviors for $g_l(\lambda)$ as $\lambda \to -\infty$ are also required.

As in the case of $g = sl_2$, the pairing $(\ ,\ ) : \mathcal{F}^*(\eta) \otimes \mathcal{F}(\eta) \to \mathbb{C}$ between the left and right Fock spaces can be uniquely defined by the following prescriptions,

- $(\langle \text{vac}|, |\text{vac}\rangle) = 1,$
- $$\left(\langle \text{vac}| \int_0^{+\infty} d\lambda g(\lambda) a_i(\lambda), \int_{-\infty}^{0} d\mu f(\mu) a_j(\mu) |\text{vac}\rangle \right)$$ $$= \int_C \frac{d\lambda \ln(-\lambda)}{2\pi i} g(\lambda) f(-\lambda) \alpha_{ij}(\lambda),$$
- the Wick theorem.

Now let the vacuum states $|\text{vac}\rangle$ and $\langle \text{vac}|$ be such that

$$a_i(\lambda)|\text{vac}\rangle = 0, \quad \lambda > 0, \quad P_i|\text{vac}\rangle = 0,$$
$$\langle \text{vac}|a_i(\lambda) = 0, \quad \lambda < 0, \quad \langle \text{vac}|Q_i = 0.$$

Let $f(\lambda)$ be analytic in some neighborhood of the real λ-line, satisfying proper analytic behaviors as $\lambda \to \pm\infty$, and also have simple poles at $\lambda = 0$. Then, the action of expressions like

$$F = \exp\left(\int_{-\infty}^{+\infty} d\lambda f(\lambda) a_i(\lambda) \right)$$

on $\mathcal{F}(\eta)$ and $\mathcal{F}^*(\eta)$ are given, respectively, by the decompositions $F = F_- F_+$ and $F = \tilde{F}_- \tilde{F}_+$, where

$$F_- = \exp\left(\int_{-\infty}^{0} d\lambda f(\lambda) a_i(\lambda) \right),$$

$$F_+ = \lim_{\epsilon \to 0^+} e^{\epsilon \ln \epsilon f(\epsilon) a_i(\epsilon)} \exp\left(\int_{\epsilon}^{+\infty} d\lambda f(\lambda) a_i(\lambda) \right),$$

$$\tilde{F}_- = \lim_{\epsilon \to 0^+} e^{\epsilon \ln \epsilon f(-\epsilon) a_i(-\epsilon)} \exp\left(\int_{-\infty}^{-\epsilon} d\lambda f(\lambda) a_i(\lambda) \right),$$

$$\tilde{F}_+ = \exp\left(\int_{0}^{+\infty} d\lambda f(\lambda) a_i(\lambda) \right).$$

Moreover, these two actions are adjoint to each other, and the product of normal ordered operators like F atisfy our normal ordering rule (30). This completes the description of Fock spaces at level 1.

The Fock spaces for level k bosonic representation of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ is nothing but the direct product of k copies of the level 1 Fock spaces, namely $\mathcal{F}^{(k)}(\eta^{(0)}, \ldots, \eta^{(k-1)}) = \mathcal{F}(\eta^{(0)}) \otimes \cdots \times \mathcal{F}(\eta^{(k-1)})$. The left Fock space for the level k bosonic representation has a similar structure, $\mathcal{F}^{*(k)}(\eta^{(0)}, \ldots, \eta^{(k-1)} = \mathcal{F}^*(\eta^{(0)}) \otimes \cdots \mathcal{F}^*(\eta^{(-1)})$.

3.4. The case of $c = 0$: Evaluation representation

As mentioned earlier, the structure of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ changes drastically from $c \neq 0$ to $c = 0$. This change is not only reflected in the different asymptotic behaviors for the generating currents, but also in the trivialization of the structure of the infinite Hopf family (see Remark 3), and it also affects the representation theory at $c = 0$.

Just like the usual affine Lie algebras and the q-affine algebras, among the class of level 0 representations for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$, there is a special subclass which is finite dimensional. We adopt the terminology from the representation theory of affine and q-affine algebras and call the finite dimensional level 0 representations of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ the evaluation representations.

Recall that there is no difference between the algebras $\mathcal{A}_{\hbar,\eta^{(m)}}(\hat{g})_0$ and $\mathcal{A}_{\hbar,\eta^{(n)}}(\hat{g})_0$ for different n and m. Recall also that the evaluation representations for the usual q-affine algebras are best written in terms of "half currents" rather than the "total currents" $E_i(u)$ and $F_i(u)$ which we have been using for $\mathcal{A}_{\hbar,\eta}(\hat{g})$ so far. Therefore, it seems that the first step to give an evaluation representation for the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ is to split the total currents $E_i(u)$ and $F_i(u)$ into half currents. This task can be fulfilled in a completely analogous way as in the sl_2 case.

We define (for generic c) the half currents $e_i^{\pm}(u)$ and $f_i^{\pm}(u)$ as follows:

$$e_i^{\pm}(u) = \pi\eta \int_{C_\pm} \frac{\mathrm{d}v}{2\pi \mathrm{i}} \frac{E_i(v)}{\sinh \pi\eta(u - v \pm \mathrm{i}\hbar c/4)},$$

$$f_i^{\pm}(u) = \pi\eta' \int_{C_\mp} \frac{\mathrm{d}v}{2\pi \mathrm{i}} \frac{F_i(v)}{\sinh \pi\eta'(u - v \mp \mathrm{i}\hbar c/4)},$$

where the contours $C_\pm$ run from $-\infty$ to $+\infty$ along the real axis, with the points $u + \mathrm{i}\hbar c/4 + \mathrm{i}k/\eta (k \geq 0)$ above C_+, $u + \mathrm{i}\hbar c/4 + \mathrm{i}k/\eta (k < 0)$ below C_+, $u - \mathrm{i}\hbar c/4 + \mathrm{i}k/\eta' (k > 0)$ above C_-, $u - \mathrm{i}\hbar c/4 + \mathrm{i}k/\eta' (k \leq 0)$ below C_-.

Note that the above choice of integral contours imply that the half currents $e_i^+(u)$, $f_i^+(u)$ are properly defined only in the strip

$$\Pi_+ = \left\{ -\frac{1}{\eta} - \frac{\hbar c}{4} < \operatorname{Im} u < -\frac{\hbar c}{4} \right\},$$

whilst $e_i^-(u)$, $f_i^-(u)$ are defined only in the strip

$$\Pi_- = \left\{ \frac{\hbar c}{4} < \operatorname{Im} u \frac{\hbar c}{4} + \frac{1}{\eta} \right\}.$$

We thus call the strips $\Pi_\pm$ the "domains of analyticity" for the half currents. We remark that the half currents satisfy the following Ding–Frenkel-like relations,

$$e_i^+\left(u-\frac{\mathrm{i}\hbar c}{4}\right)-e_i^-\left(u+\frac{\mathrm{i}\hbar c}{4}\right)=E_i(u),$$
$$f_i^+\left(u+\frac{\mathrm{i}\hbar c}{4}\right)-f_i^-\left(u-\frac{\mathrm{i}\hbar c}{4}\right)=F_i(u),$$

however, these relations should be understood in some proper "analytic continuation sense" in contrast to the direct decompositions of formal power series [20,22] because in the above equations, $e_i^+(u-\mathrm{i}\hbar c/4)$ and $e_i^-(u+\mathrm{i}\hbar c/4)$ cannot be properly defined simultaneously for the same value of u.

Following the same spirit of analytical continuation, we have

$$e_i^-(u)=-e_i^+(u-\mathrm{i}/\eta''),\quad f_i^-(u)=-f_-^+(u-\mathrm{i}/\eta'')$$

and

$$F_i^+(u)=H_i^+(u-\mathrm{i}/\eta''),$$

where $u\in\Pi_-$.

The following proposition gives a simplest evaluation representation for $\mathcal{A}_{\hbar,\eta}(\hat{g})$ with $g=sl_{r+1}$.

Proposition 5. *Let V be an $(r+1)$-dimensional vector space with orthogonal basis $\{v_0, v_1, \ldots, v_r\}$. The $(r+1)$-dimensional evaluation representation of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ with $g=sl_{r+1}$ on $V_z(\eta)=V\otimes\mathbb{C}[[\mathrm{e}^{\pi\eta z}]]$ is given by the following actions ($u\in\Pi_+$),*

$$\begin{aligned}
e_l^+(u)v_{j,z}&=\delta_{lj}\frac{-\sinh \mathrm{i}\pi\eta\hbar}{\sinh \pi\eta(u-z-(r-l)/2\,\mathrm{i}\hbar)}v_{j-1,z},\\
f_l^+(u)v_{j-1,z}&=\delta_{lj}\frac{-\sinh \mathrm{i}\pi\eta\hbar}{\sinh \pi\eta(u-z-(r-l)/2\,\mathrm{i}\hbar)}v_{j,z},\\
H_l^+(u)v_{j,z}&=\delta_{lj}\frac{\sinh \pi\eta(u-z-(r-l-2)/2\mathrm{i}\hbar)}{\sinh \pi\eta(u-z-(r-l)/2\mathrm{i}\hbar)}v_{j,z}\\
&\quad+\delta_{l-1,j}\frac{\sinh \pi\eta(u-z-(r-l+2)/2\,\mathrm{i}\hbar)}{\sinh \pi\eta(u-z-(r-l)/2\,\mathrm{i}\hbar)}v_{v,z}\\
&\quad+(1-\delta_{lj}-\delta_{l-1,j})v_{j,z}.
\end{aligned}$$

The relations for the "negative" half currents are given by the same formulas but with $u\in\Pi_-$.

Note that for $r=1$, the above evaluation representation reduces to the one presented in [24] for $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$; for $\eta\to 0$, it reduces to the $(r+1)$-dimensional evaluation representation for $DY_\hbar(sl_{r+1})$ [20].

3.5. The intertwining (vertex) operators

One of the important ingredients in the representation theory of affine algebras is the intertwining operators which intertwine the infinite-dimensional representation and its tensor product with evaluation representation. For the infinite Hopf family of algebras, we can define analogous objects, also called intertwining operators, acting on the space of tensor product of the infinite-dimensional representation of the subsequent member of the same family, or on the space of tensor product of the evaluation representation and some infinite-dimensional representation of a fixed member of the family.

Taking as the infinite-dimensional representation the level 1 bosonic representation, as the evaluation representation the $(r+1)$-dimensional representation obtained above for $g = sl_{r+1}$, we now proceed to give the definition of a particular set of intertwining operators.

Definition 3. The intertwining operators (vertex operators) (here $\eta' = 1/(\hbar + 1/\eta)$)

$$\Phi(z) : \mathcal{F}(\eta) \to \mathcal{F}(\eta) \otimes V_z(\eta'), \qquad \Phi^*(z) : \mathcal{F}(\eta) \otimes V_z(\eta') \to \mathcal{F}(\eta),$$

$$\Psi^*(z) : V_z \otimes \mathcal{F}(\eta) \to \mathcal{F}(\eta), \qquad \Psi(z) : \mathcal{F}(\eta) \to V_z \otimes \mathcal{F}(\eta)$$

are those commuting with the action of $\mathcal{A}_{\hbar,\eta}(\hat{g})$, [3]

$$\Phi(z)x = \Delta(x)\Phi(z), \qquad \Phi^*(z)\Delta(x) = x\Phi^*(z),$$

$$\Psi^*(z)\Delta(x) = x\Phi^*(z), \qquad \Psi(z)x = \Delta(x)\Psi(z),$$

where $x \in \mathcal{A}_{\hbar,\eta}(\hat{g})$.

The components of these vertex operators are defined as follows:

$$\Phi(z)v = \sum_{j=0}^{r} \Phi_j(z)v \otimes v_j, \qquad \Phi^*(z)(v \otimes v_j) = \Phi_j^*(z)v,$$

$$\Psi^*(z)(v_j \otimes v) = \Psi_j^*(z)v, \qquad \Psi(z)v = \sum_{j=0}^{r} v_j \otimes \Psi_j(z)v.$$

where $v \in \mathcal{F}(\eta)$ and $v_j \in V$.

Using the explicit form of the evaluation representation given in the last subsection and the comultiplication formulas (19)–(28), we are ready to obtain the intertwining relations (the commutation relations between vertex operators and the generating currents for $\mathcal{A}_{\hbar,\eta}(\hat{g})$). The result will be a rather long list and we omit it here. We note that similar constructions for q-affine algebras were made in [8].

[3] In [24], a twisted version of the vertex operators was defined so that the commutation relations of the twisted vertex operators yield the two-body S-matrix for the sine–Gordon model. In our case, we do not have such motivations to define twisted vertex operators. Moreover, remember that the comultiplications of [24] are different from the one we are using.

Using the bosonic Heisenberg algebra $\mathcal{H}(\eta)$, one can, in principle, obtain bosonic realizations of these intertwining operators. Then the calculation for the commutation relations between these intertwining operators and the correlation functions of such operators will become possible. We leave such tasks to future studies.

4. Discussion

In closing this paper we give the conclusions and some discussion.

We defined the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$ and its infinite Hopf family for all simply laced Lie algebras g. Using the deformed Heisenberg algebra $\mathcal{H}(\eta)$, we obtained the level 1 bosonic representation, and then by repeated use of the comultiplication we got the representations for all positive integer levels. For $g = sl_{r+1}$, we also gave the simplest $(r+1)$-dimensional evaluation representation and the intertwining relations for the level 1 representation and the $(r+1)$-dimensional evaluation representation.

Clearly, many relevant problems are still left open, among which we mention several which we would like to solve in future works.

The first problem is: Why not nonsimply laced Lie algebras g? Indeed, no reason can be stated a priori that no analogous algebras exist for nonsimply laced algebras g. However, for self-consistency, we intentionally excluded nonsimply laced g in our consideration. The reason is that, for such a g, the Cartan matrix is *not* symmetric, so that the Heisenberg algebra $\mathcal{H}(\eta)$ is not well-defined (the condition (38) is violated). Probably, the way around is to use the *symmetrized* Cartan matrix instead of the Cartan matrix. Then, we can give well-defined Heisenberg algebra $\mathcal{H}(\eta)$, but the Serre-like relations (8) and (9) are still not enough to define the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$, because there are cases for $A_{ij} = -2, -3$, etc.

The second problem is the other possible realizations of the algebra $\mathcal{A}_{\hbar,\eta}(\hat{g})$. For q-affine algebras, Yangian doubles and $\mathcal{A}_{\hbar,\eta}(\widehat{sl}_2)$, three different realizations are known to exist, i.e. the current realization, Drinfeld generator realization and the Reshetkhin–Semenov–Tian–Shansky (*RLL*) realization. For our algebra, it seems important to find the third realization because this realization has direct connection with the Yang–Baxter relation and, hence, is more convenient while considering the possible application of the algebra in integrable quantum field theories.

As mentioned in Section 1, we postulate that our algebra might have important application in describing the quantum symmetries of affine Toda theory, however such applications can be made possible only if we have identified the R-matrix of our algebra with the quantum S-matrix of affine Toda theory. In this respect, the other form of the comultiplication which is compatible with *RLL* relations is also important because, under such a comultiplication, the commutation relations between the interwining operators would become a set of Faddeev–Zamolodchikov-like algebra which should be explained as the operator form of the quantum scattering of the corresponding integrable quantum field theory – the affine Toda theory as we postulate it.

Various considerations on the different choices of domains for the deformation parameters $\hbar$ and η are also important. On this point, the authors of [24] have already listed many problems with which we wholeheartedly agree. Besides the problems listed in [24], we are also interested in the case $\hbar \to \infty$, which should correspond to the case of crystal base for q-affine algebras.

Lastly, we would like to mention the possible connections between our algebra and the quantum $(\hbar, \xi)$-deformed Virasoro and W-algebras. The q- and $\hbar$-deformed Virasoro (and W) Poisson algebras were known to be closely connected to q-affine algebra and Yangian double at the critical level [7,16]. The quantum versions of these deformed algebras were also known to exist and nobody knowns to which deformed affine algebras they correspond The algebras given in this paper may be the right candidate to correspond to the quantum $(\hbar, \xi)$-deformed Virasoro W-algebras. We point out that algebras, corresponding to the q, p-deformed quantum W-algebras also exist, which are generalizations of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ to other g with higher ranks. We shall present the bosonic representation for the current realization for such algebras (which we call $\mathcal{A}_{q,p}(\hat{g})$, representative of yet another example of infinite Hopf family structure) in the next paper [19].

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